Pensions and Unemployment: Math Sheet*

Christian Keuschnigg
University of St.Gallen (IFF), CEPR and CESifo

Mirela Keuschnigg
Johannes Kepler University Linz and University of St.Gallen

This version: December 13, 2003

Abstract

This math sheet sets out a stylized core model explaining labor market effects of pension reform in the presence of search unemployment.

Keywords:Pension reform, search, unemployment.
JEL-Classification: C68, D58, E62, J64,H55 ????

*This technical note is part of a larger research project on "Public Finance, Unemployment and Growth". Financial support by the FWF, project no. P14702, is gratefully acknowledged.

1 A Stylized Model

1.1 Households

- OLG with endogenous labor supply, extensive (search) and intensive (hours worked).
- Timing: 1. Job search of workers (endogenous search intensity ζ_t) and of firms (vacancies v_t, job creation). 2. Matching, splitting agents into employed and unemployed. 3. Wage bargaining. Steps 1-3 occur at beginning of period one. 4. Labor supply (hours worked e_t) and production. 5. Life-cycle consumption in first and second period.
- Basic assumptions: Ricardian production technology with fixed labor productivity. Agents are risk neutral. Consumption is perfectly substitutable, only present value is relevant. The interest factor R is thus equal to the time preference rate and constant.
- Preferences: expected life-time utility of an agent born in period t is

$$U_t = E\left[\bar{C}_t^i - \varphi\left(e_t^i\right)\right] - \psi\left(\zeta_t\right), \quad \bar{C}_t^i \equiv C_t^{1i} + C_t^{2i}/R. \tag{1}$$

- Preferences are separable and linear in life-time consumption. Search ζ_t occurs prior to effort e_t^i. After matching, the agent may be either employed or unemployed, $i \in \{E, U\}$. We will normalize work effort of an unemployment to zero, denoting by e and $\varphi(e)$ the hours worked and disutility of effort of an employed agent. Prior to matching, expected utility must be taken over welfare V^i derived in different labor market states. Expected utility also depends on search intensity ζ which determines the probability of finding a job, rather than staying unemployed.
- **Backward solution:** solve first for consumption (stage 5). After matching, agents earn income y^i, depending on the labor market state. Given perfect intertemporal substitution, agents do not care about when to consume (i.e. savings are not well

determined). Welfare in stage 5 is thus equal to life-time consumption. Taking income as given,

$$S_t^i = y_t^i - C_t^{1,i}, \quad C_{t+1}^{2,i} = RS_t^i + E_{t+1}^i \quad \Rightarrow \quad \bar{C}_t^i = C_t^{1,i} + \frac{C_{t+1}^{2,i}}{R} = y_t^i + \frac{E_{t+1}^i}{R}. \tag{2}$$

- If employed, the agent earns a net wage but must pay contributions to the PAYG system. The statutory rate is t. If unemployed, the agent earns a fixed non-market income z (home production).[1] When old, agents receive a pension from the PAYG system. The pension of an employed is earnings related with a replacement rate ϕ while an unemployed agent gets a minimum pension $\bar{E}$:

$$y_t^E = (1-t)\, w_t e_t, \quad y_t^U = z, \quad E_{t+1}^E = \phi_{t+1} \cdot (1-t)\, e_t w_t, \quad E_{t+1}^U = \bar{E}. \tag{3}$$

- Substituting (3) into (2) and (1) gives life-time welfare from consumption. Substituting the pension rule defines the implicit tax component $\tau < t$. The implicit tax shows to what extent the worker perceives PAYG contributions as a tax:

$$\begin{aligned} \bar{C}_t^E &= (1-t)\, w_t e_t + \phi_{t+1} \cdot e_t w_t / R = (1-\tau)\, w_t e_t, \\ \tau &\equiv t - \tfrac{(1-t)\phi_{t+1}}{R} < t, \quad \tfrac{\partial \tau}{\partial t} = 1 + \tfrac{\phi_{t+1}}{R}, \quad \tfrac{\partial \tau_t}{\partial \phi_{t+1}} = -\tfrac{1-t}{R}. \end{aligned} \tag{4}$$

- Depending on the agent's labor market state, life-time welfare of an agent net of effort cost is $V^i = \bar{C}^i - \varphi(e^i)$. Since effort cost of an unemployed is zero, we have

$$V_t^E = (1-\tau)\, w_t e_t - \varphi(e_t), \qquad V_t^U = z + \bar{E}/R. \tag{5}$$

- **Stage 4 (hours worked):** the intensive labor supply decision is to choose the hours worked, or choosing work effort. At this stage, search effort ζ is already sunk so that the disutility ψ of search can be ignored. Maximizing remaining welfare in (1) is equivalent to maximizing (5),[2]

$$V_t^E = \max_e \{(1-\tau)\, w_t e_t - \varphi(e_t)\} \quad \Rightarrow \quad \left(1-\tau^i\right) w_t = \varphi'(e_t). \tag{6}$$

[1] This may be later replaced by a government financed unemployment benefit, if it doesn't become too complicated.

[2] For computational purposes, we choose an isoelastic specification of effort costs, $\varphi(e) = e^{1+1/\varepsilon}/(1+1/\varepsilon)$, giving $e = [(1-\tau)\, w]^{\varepsilon}$. In this case, the labor supply elasticity is constant.

- Applying the envelope theorem, we show how the shadow value of work changes with wages and policy parameters,

$$\frac{\partial V^E}{\partial w} = (1-\tau)\, e > 0, \qquad \frac{\partial V^E}{\partial \tau} = -we < 0, \tag{7}$$

  where the implicit tax rate changes as indicated in (4).

- **Stage 3 (wage negotiation):** This will be explained in subsection 2.3 below.

- **Stage 2 (matching):** Matching determines the probability ζf with which the agent is able to locate a job. With probability $1-\zeta f$, the search effort ζ is unsuccessful, leaving the worker unemployed. This probability partly depends on labor market tightness which fixes the component f beyond the worker's control. But higher individual search effort ζ raises the individual chance to locate a job. The matching probabilities are explained in subsection 2.4 below.

- **Stage 1 (job search):** Anticipating the results of matching and the subsequently derived welfare, (the risk neutral) agents choose search intensity to maximize expected life-time welfare in (1),[3]

$$U = \max_{\zeta} \left\{ \zeta f \cdot V^E + (1-\zeta f) \cdot V^E - \psi(\zeta) \right\} \quad \Rightarrow \quad \left(V^E - V^U \right) \cdot f = \psi'(\zeta). \tag{8}$$

1.2 Firms

- Ricardian technology with fixed labor productivity F_L. To recruit workers, firms must search on the labor market by posting a sufficiently large number of v vacancies. Search cost κ per vacancy. A vacancy can be filled with a suitable worker with probability q, see subsection 4 below. With probability $1-q$, search is unsuccessful and the search cost is lost.

[3] Again we may conveniently choose an isoelastic specification of search costs, $\psi(\zeta) = \zeta^{1+1/\omega}/(1+1/\omega)$, giving $\zeta = \left[\left(V^E - V^U \right) f \right]^{\omega}$, so that the elasticity of search effort with respect to expected returns to search is a constant ω.

- Once the job is filled, a wage is negotiated and production starts, yielding a job rent J per worker. Investment takes place at the beginning of period if the expected job rent at least covers the search cost, $J \cdot q \geq \kappa R$, where $R\kappa$ is the "user cost" of vacancies, including foregone interest. Firms post vacancies until they break even. The number of vacancies is then determined in equilibrium. Free entry leads to

$$J \cdot q = \kappa R, \qquad J \equiv (F_L - w)\, e. \tag{9}$$

- The value of a filled job falls with a higher wage.

1.3 Wage Bargaining

- After matching, the firm and worker have to bargain a wage to share the joint surplus $V^E - V^U + J$ that is created by the match. The worker's surplus is the excess value of employment over her outside option V^U. The outside option of the firm is zero. With μ denoting the worker's bargaining power, Nash wage bargaining determines the wage.

$$\Omega(i) = \max_{w(i)} \left[V^E(i) - V^U\right]^{\mu} \cdot [J(i)]^{1-\mu}. \tag{10}$$

- When the worker and firm break up, each side is left with the fallback position or outside values V^U and $J = 0$ which are independent of the wage negotiated in relationship i. When the pair agrees on a higher wage, the worker's value increases as in (7). The firm's job rent, in contrast, declines by $\partial J/\partial w = -e$. The last equation reflects our assumption that the firm takes effort e as given and not to be influenced by the wage rate (no efficiency wage effect).[4]

- Taking account of these derivatives, one obtains the f.o.c. (symmetry of all pairs i is already imposed)

$$\mu(1-\tau)\, J = (1-\mu)\left(V^E - V^U\right). \tag{11}$$

[4] We should try to derive the wage equation (12) by taking the motivation effect de/dw into account, i.e. we should use the derivative stated in (16) and see whether this gives a manageable wage equation.

- After substituting for J and V^E, some rearrangements lead to the wage equation

$$w = \mu F_L + (1-\mu)\,\frac{\varphi(e) + V^U}{(1-\tau)\,e}. \tag{12}$$

- The wage is only implicitely determined since hours worked depend on the wage as well. To illustrate this interaction, we define the wage equation $w(e)$, and compare it with the labor supply schedule $e(w)$ as in (6). To characterize the wage equation, differentiate (12) w.r.t to e and get a slope [use also the f.o.c. in (6) and the definition of V^E] which is positive on account of $V^E > V^U$ (see 7):[5]

$$\frac{dw}{de} = \frac{(1-\mu)\left(V^E - V^U\right)}{(1-\tau)\,e^2} > 0 \tag{13}$$

- The wage curve is drawn in Figure 1 as a function of hourse worked, $w(e)$. It shifts to the right on account of a higher outside option V^U, and also with a higher implicit social security tax τ. The latter is the usual tax shifting result in the face of a constant outside option. For stability reasons, the wage curve must be steeper than the labor supply curve $e(w)$ given by (6). The labor supply curve shifts down as the implicit tax increases.

Figure 1: Wage and Labor Supply

- From graphical analysis, we obtain the results

$$w\left(\underset{+}{\tau}, \underset{+}{V^U}\right), \qquad e\left(\underset{-}{\tau}, \underset{+}{V^U}\right). \tag{14}$$

1.4 Matching

- The working population has constant mass one. A fraction ζf is employed while a fraction $1-\zeta f$ remains unemployed. The number of matches $m(\zeta, v)$ is a linear homogeneous function of workers' total search input ζ and the firms total search

[5] Using the isoelastic specification of effort, we may also write the wage as $w = \mu F_L + (1-\mu)\left(\frac{w}{1+1/\varepsilon} + \frac{V^U}{(1-\tau)e}\right)$.

input v. Only a fraction f of total worker search attempts are successful, and only a fraction q of posted vacancies can be filled. The following identities must hold,

$$\zeta \cdot f = m(\zeta, v) = q \cdot v. \tag{15}$$

- If $m(\zeta, v)$ is linear homogeneous, the transition rates depend on labor market tightness θ as follows,

$$f'(\theta) > 0, \qquad q'(\theta) < 0, \qquad \theta q(\theta) = f(\theta), \qquad \theta \equiv v/\zeta. \tag{16}$$

- If we assume a Cobb Douglas specification, $m = \zeta^{\eta} v^{1-\eta}$, we have $f(\theta) = \theta^{1-\eta}$, $q(\theta) = \theta^{-\eta}$, and obviously $f(\theta) = \theta q(\theta)$.

1.5 Labor Market Tightness

- Search investments depend on the rents that workers and firms can appropriate when entering the employment relationship. Reformulating the bargaining condition (11) yields

$$\left(V^E - V^U\right) = \frac{\mu}{1-\mu}(1-\tau) \cdot J(w), \qquad \frac{\partial J}{\partial w} = -e \cdot \left(1 - \frac{F_L - w}{w}\varepsilon\right) < 0. \tag{17}$$

- While the firm takes effort as given in the wage negotiation, the wage will in fact influence effort of the worker subsequently.[6] This must be taken into account when calculating the equilibrium change in the job rent J. The sign in (16) assumes that the wage reduces job rent, even though it stimulates value increasing work effort. The latter motivation effect is assumed not to dominate the direct effect of the wage.

- Now we can compute how the return to search depends on labor market tightness θ by considering the search investment conditions, (8) and (9). We repeat,

$$\left(V^E - V^U\right) \cdot f(\theta) = \psi'(\zeta), \qquad J(w) \cdot q(\theta) = \kappa R. \tag{18}$$

[6] We should definitely try to take the motivation effect de/dw into account when deriving the wage equation in (12), see the remark after (10).

- Anticipating the results of wage negotiations as in (14) and noting $\partial J/\partial w < 0$, the implicit tax and the worker's outside option both diminish the job rent of the firm, $J\left(\underset{-}{\tau}, \underset{-}{V^U}\right)$. The firm's investment (or free entry) condition in (17) thus determines, in equilibrium, a unique value of labor market tightness which is the vertical job creation schedule in Figure 2. It shifts to the left when the implicit social security tax or the worker's outside option increase.

Figure 2: Market Tightness and Employment

- Wage bargaining uniquely relates the worker's expected return to search, $\left[V^E - V^U\right] \cdot f(\theta)$, to the firm's job rent as in (17). It is therefore clear and very intuitive that the tax and the outside option, τ and V^U, both reduce the returns to search $V^E - V^U$ and thus lead to lower search intensity. On the other hand, a tighter labor market improves the workers' chances to locate a job, $f'(\theta) > 0$. Summarizing, search intensity increases with θ and falls with the tax and the outside option, $\zeta\left(\underset{+}{\theta}; \underset{-}{\tau}, \underset{-}{V^U}\right)$. Figure 2 illustrates the upward sloping schedule of extensive labor supply $L = \zeta \cdot f$.

- Figure 2 shows that a higher, implicit social security tax τ and more generous basic pension to previously unemployed V^U reduce labor market tightness and equilibrium employment.

- Summary: Implicit social security tax τ: boosts wages (wage shifting), reduces "net" wage $(1-\tau)\,w$, reduces intensive and extensive labor supply (e and ζ), reduces labor market tightness θ, and therefore aggregate employment $L = \zeta f(\theta)$, while unemployment $1 - L = 1 - \zeta f$ increases. Aggregate output or national income $Y = F_L e \cdot L + z \cdot (1 - L)$ falls even more than employment due to the negative intensive labor supply effect. Same qualitative effects are immediately derived for the basic pension $\bar{E}$ to the unemployed.

1.6 Pension System

- For the moment, assume that there is no government debt. The public budget is

$$t \cdot we \cdot \zeta_t f_t = E \cdot \zeta_{t-1} f_{t-1} + \bar{E} \cdot \left(1 - \zeta_{t-1} f_{t-1}\right). \tag{19}$$

- Pension payments to the employed are wage related, $E_t = \phi_t \cdot (1-t) w_{t-1} e_{t-1}$. In a steady-state, we have

$$[t - \phi \cdot (1-t)] \cdot we \cdot \zeta f = \bar{E} \cdot (1 - \zeta f). \tag{20}$$

- If there were no pension, $\bar{E} = 0$, the statutory tax rate t is related to the replacement rate as $t = \phi / (1 + \phi)$. If the system must also finance a basic pension $\bar{E}$ to non-contributors, then the statutory contribution rate must be accordingly higher.

Appendix

- Walras' Law: This is just for completeness. Since present and future consumption is perfectly substitutable, savings is not separately determined. Agents save whatever is required to finance the investment cost of job creation, and consume the rest. We show Walras' Law only to check consistency of the model formulation.

- The private budget identities are $S^E = (1-t)\, we - C^{1,E}$ and $C^{2,E}_{t+1} = RS^E_t + E_{t+1}$ for the employed and $S^U = z - C^{1,U}$ and $C^{2,U}_{t+1} = RS^U_t + \bar{E}_{t+1}$. Define aggregate household sector variables as

$$C^1 = \zeta f \cdot C^{1,E} + (1 - \zeta f) \cdot C^{1,U}. \tag{A.1}$$

- We therefore get the aggregate budget identities

$$\begin{aligned} S &= (1-t)\, we \cdot \zeta f + z \cdot (1 - \zeta f) - C^1, \\ C^2_t &= \zeta f_{t-1} \cdot \left[RS^E_{t-1} + E_t\right] + (1 - \zeta f_{t-1}) \cdot \left[RS^U_{t-1} + \bar{E}_t\right]. \end{aligned} \tag{A.2}$$

- Adding up, we find aggregate consumption, $C_t \equiv C_t^1 + C_t^2$, equal to

$$\begin{aligned} C_t \;\equiv\; & (1-t)\, we \cdot \zeta f + z \cdot (1-\zeta f) - S \\ & + RS_{t-1} + \zeta f_{t-1} \cdot E_t + (1-\zeta f_{t-1}) \cdot \bar{E}_t. \end{aligned} \tag{A.3}$$

- Now we can substitute the GBC to get aggregate consumption

$$C_t = we \cdot \zeta f + z \cdot (1-\zeta f) - S_t + RS_{t-1}. \tag{A.4}$$

- Next, we note that savings of the previous period must buy the assets issued at the beginning of period t (capital market equilibrium),

$$\kappa v_t = S_{t-1}, \qquad (F_L - w)\, e \cdot qv = R\kappa v. \tag{A.5}$$

  The second equation repeats the firms break even condition in (9).

- Sustitute this into (A.4) and use the matching flow relation in (14) where $m = L$ denotes the number of productive matches equal to aggregate employment. This yields the GDP identity which proves Walras' Law:

$$C_t + \kappa v_{t+1} = Y \equiv F_L e \cdot L + z \cdot (1-L)\,. \tag{A.6}$$

Figures

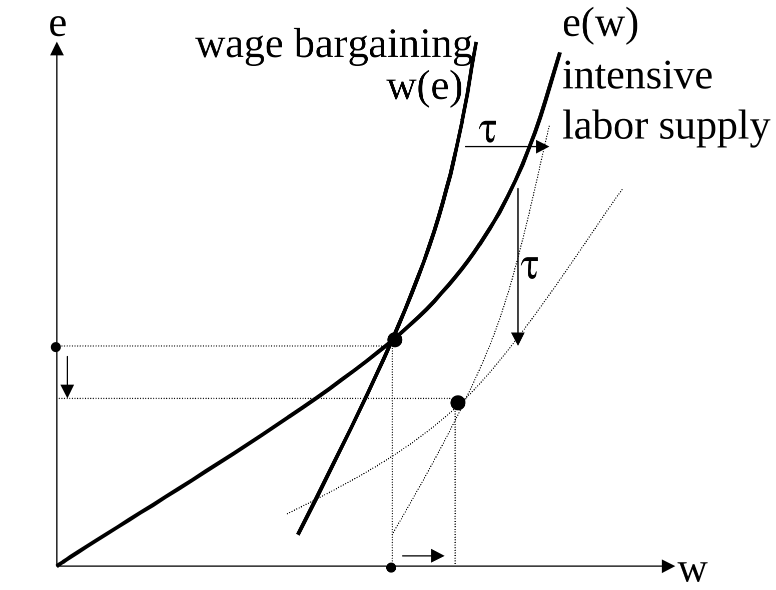

Fig.1: Wage and Labor Supply

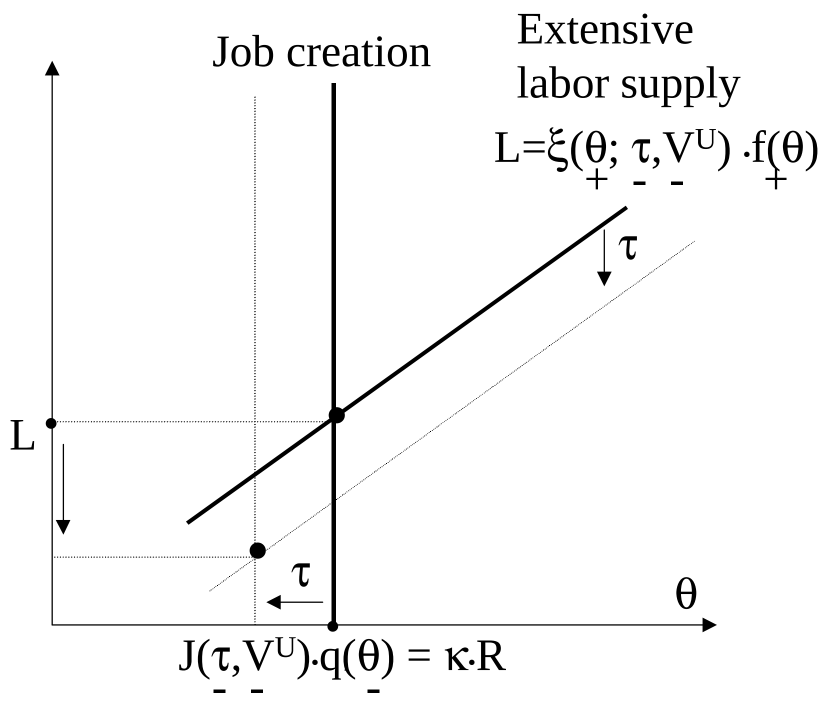

Fig.2: Market Tightness & Employment